2017-61

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Testing Goodwin with a Stochastic Differential Approach The United States (1948 - 2017)

Florent Mc ISAAC *

November 2017

Please cite this paper as: Mc ISAAC, F. (2017), "Testing Goodwin with a Stochastic Differential Approach–The United States (1948-2017)", *AFD Research Papers Series*, No. 2017-61, November.

Contact at AFD: Florent Mc Isaac(mcisaacf@afd.fr)

* Agence Française de Développement

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ISSN 2492 - 2846

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Florent Mc Isaac, Agence Française de Développement

Abstract

This paper follows Harvie (2000)'s research program in testing both Goodwin (1967)'s predator-prey model and the extension proposed by van der Ploeg (1985). The author's aim is to provide a guideline for the estimation and the backtesting strategy that can be applied to such a class of continuous-time macroeconomic model. The goal of this paper is to propose and test stochastic differential equations for Goodwin's model and one of its extension by using an estimation technique based on simulated maximum likelihood developed by Durham and Gallant (2002). The data considered here is that of wage share and employment rate in the United States from 1948:Q1 to 2017:Q2. Results show that two structural breaks–in the beginning of the 80s' and late 90s'–are likely to have occurred and Goodwin-type model endowed with Leontief production technology explains more accurately the data than the van der Ploeg's CES production function. These results are partly confirmed by a backtesting strategy which highlights that the forecasting property of the Goodwin model is overwhelmingly superior to a VAR model on the considered data, especially for the CES specification. Both the estimation and backtesting strategies can be used to assess the empirical improvement on any extension of the Goodwin model.

Key words: Simulated maximum likelihood, Lodka-Volterra; Stochastic Differential Equation; Goodwin; Dynamical systems; Backtesting

JEL classification: C15, E30, J20, E11

Acknowledgments
This work benefited from the support of the Chair Energy and Prosperity, Energy Transition Financing and Evaluations, Fondation du Risque, ADEME, Air Liquide Advanced Technologies, AFD, CDC Climat, SNCF, Schneider Electric and Mirova. I am grateful to Gaël Giraud, Matheus Grasselli, Adrien Nguyen-Huu, Christophe Chorro, Ivar Ekeland, Marc Lavoie, and participants of the "1st International Conference on Econometrics and Statistic" in Hong Kong (June, 2017) for their generous comments. All errors are, of course, my own.

Original version: English

Accepted: November 2017

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Florent Mc Isaac†

November 17, 2017

Abstract

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†Author's email: mcisaacf@afd.fr. Institutions: AFD, Agence Française de Développement; chair Energy and Prosperity.

1 Introduction

It has been half a century since Goodwin (1967) developed a model of endogenous real growth cycles. Based on a simple and well known dynamic–the nonlinear Lotka-Volterra prey-predator model–Goodwin's model appeals in its simplicity and can be easily applied by a wide range of researchers in a variety of fields (physics, biology among others). In the late 1970s and 1980s, this research focused on relaxing one or more of the original model's assumptions and on adding new variables.[1] More recently, with the development of fast computing machines, which lower the costs of numerical simulation of continuous-time models, a large body of literature, especially in higher dimension, has emerged.[2] Although the literature is enriched by new theoretical extensions the empirical development has not been extensively explored. Currently, the best-known empirical research on Goodwin's model is perhaps that published by Harvie (2000).

In the year 2000, Harvie published a paper with mixed conclusions. On the one hand, using qualitative evaluation, Harvie acknowledged that the Goodwin model makes clear predictions on the interdependence of the employment rate and income distribution based on the clockwise behavior of the data over ten OECD countries. On the other hand, Harvie's findings for empirical estimation of the equilibrium point (the growth cycles' centers) and the cyclical periodicity for each country did not give satisfactory results. Harvie concluded by saying that further extension of the model should be explored in order to increase the reliability of the model's behavior. However, Grasselli & Maheshwari (2017b) showed that Harvie made reporting errors for the short term Phillips curve coefficients, thus destabilizing the conclusions. Furthermore, the findings of Grasselli & Maheshwari (2017a) provide a more optimistic picture of the Lotka-Volterra-type model to fit empirical data, thus to explain the data's behavior.

In developing a strategy to estimate continuous-time models such as Lotka-Volerra's with low-frequency data, several potentially important caveats arise– see Section 3. Firstly, an intuitive way to tackle the estimation of such models would be to find a set of parameters that minimizes the distance of numerical deterministic simulations from the true observations.[3] Therefore, at each time, the difference between the true observation and the position of the estimated model is equal to the residual and is interpreted as being a measurement error. For example, if the observed value of the employment rate is not in the closed orbit of Goodwin's model, it is because this value has been wrongly assessed. Additionally, this type of estimation is also largely affected by the choice of the initial values. This is due to the fact that each simulation of the Goodwin model is a closed orbit,[4] thus indefinitely passing through the initial values, and therefore, the choice of the starting point will fundamentally change the outcome of the estimation. Another possible estimation strategy would be the maximum likelihood estimation. If one supposes that the Goodwin model is extended in a stochastic fashion, the system would then be made of stochastic differential equations (hereafter SDEs). Ideally, the exact transition density would be available to compute the maximum likelihood of the

[1] See Desai (1973); van der Ploeg (1985, 1987) among others.

[2] See Keen (1995); Grasselli & Costa Lima (2012); Grasselli et al. (2014); Grasselli & Nguyen-Huu (2015); Nguyen-Huu & Costa-Lima (2014); Giraud et al. (2017) among others.

[3] Numerical simulations rather than the explicit solution of the system are mentioned here since the latter is unlikely available.

[4] Indeed the model is structurally unstable, see Goodwin (1967).

model. Unfortunately, the latter is known only in a few simple cases. When the solution is unknown, one can approach it by using a first-order approximation, but the lack of high-frequency data may generate an insufficient approximation, leading to biased estimation results.

In order to overcome these problems, this article uses the technique developed by Pedersen (1995b,a) and Durham & Gallant (2002), which is commonly known as the simulated maximum likelihood estimation (hereafter SMLE). This technique is a promising alternative candidate for several reasons. First of all, it overcomes the problem of low-frequency data, since the simulated transition density converges towards the true transition density. Additionally, the estimation results are independent of the initial condition. Finally, by using SDEs rather than the deterministic counterpart, the model can explore the entire phase space, see Nguyen-Huu & Costa-Lima (2014).

After extending the Goodwin (1967) and the van der Ploeg (1985) models to a stochastic framework, this paper estimates those models using the SMLE techniques with wage share and employment data in the United States (1948:Q1-2017:Q2). A preliminary analysis on the data shows that the (parametric) relationship given by the dataset may have changed approximately in 1984:Q1 and 2000:Q1. Furthermore, I show that Leontief's extension model, as opposed to the production sector is endowed with a CES production function, is the best candidate to explain the data's behavior. A backtesting strategy based on out-of-sample error forecasts is proposed, with the aim of measuring the performance of the Goodwin model relative to a purely statistical vector autoregressive model (hereafter VAR).[5] Given the results, I show that stochastic Goodwin based models are a (very) promising alternative to a VAR model for short term forecasting purposes. Although the Goodwin model is almost uniformly superior to the VAR, further improvements might be worth exploring when forecasting the employment rate–especially in crisis period–for example, by including the investment function in the same fashion than Keen (1995) model.

This paper is organized as follows: In Section 2, an overview of the deterministic Lotka-Volterra based model and its extension made van der Ploeg (1985) is proposed, and the extension of those models to stochastic differential equations is outlined. Section 3 introduces the framework for the estimation technique and a guideline of how the identification issue is tackled. Section 4 presents the data set and the treatment assumed in the paper and an analysis of the regularity of the data, turning to the results of the estimation of stochastic Goodwin models. The backtesting strategy is treated in Section 5. Finally, Section 6 offers concluding remarks and extensions.

2 The Lotka-Volterra based models

The aim of this section is threefold: (i) to introduce the Goodwin (1967) model and its extension made by van der Ploeg (1985); (ii) to discuss Harvie's parameter estimates of the Goodwin model and; (iii) to extend those models, allowing for endogenous stochastic perturbations.

[5] VAR model, well known to be a non-economically based model and, also, for its forecasting ability, was chosen as the baseline model.

2.1 The deterministic models

Goodwin model (1967)

Goodwin (1967) introduced a growth cycle model of employment and wages based on a Lotka-Volterra predator-prey model. The predator-prey variables are the employment rate denoted by, λ, and the wage share, ω.[6] Assuming a Leontief production function, the modern version of the model[7] boils down to a two-dimensional system

$$\begin{cases} \dot{\omega} &= \omega\left(\phi(\lambda) - \alpha\right) \\ \dot{\lambda} &= \lambda\left(\frac{(1-\omega)}{\nu} - [\alpha + \beta + \delta]\right), \end{cases} \tag{1}$$

where the function, $\phi(.)$, represents a short term Phillips curve. Throughout the paper, this function is assumed the following properties to hold:

$$\phi \in \mathcal{C}^2([0,1)),\ \phi'(\lambda) > 0,\ \phi''(\lambda) \geq 0\ \forall \lambda \in [0,1),\ \phi(0) < \alpha \text{ and } \lim_{\lambda \to 1^-} \phi(\lambda) = +\infty.$$

Moreover, it is easy to verify that the function $\kappa(\omega) := (1-\omega)$ verifies the following conditions:

$$\kappa \in \mathcal{C}^2(\mathbb{R}_+),\ -\infty < \kappa'(\omega) < 0\ \forall \omega \in \mathbb{R}_+,\ \kappa(0) > \nu(\alpha + \beta + \gamma) \text{ and } \lim_{\omega \to +\infty} \kappa(\omega) = -\infty,$$

for some reliable parameters values defined shortly. These two conditions, similar to the *Assumption 1* in Nguyen-Huu & Costa-Lima (2014), are sufficient in order to ensure that, when simulation system (1), $(\omega_t, \lambda_t) \in \mathcal{D} := \mathbb{R}_+^* \times (0,1)\ \forall t \geq 0$ if $(\omega_0, \lambda_0) \in \mathcal{D}$.

Parameter estimates of the model and the values found in the literature (mostly Harvie (2000) and Grasselli & Maheshwari (2017a)) are listed below, and when necessary, methodological issues are addressed.

The productivity growth, α

The parameter $\alpha > 0$ is the labor productivity growth and drives the deterministic growth of the output-to-labor ratio, a, the underlying assumption is,

$$\frac{\dot{a}}{a} = \alpha.$$

By defining the timeserie of a as the US GDP at constant prices over the employment level, Harvie (2000) estimated the following equation,

$$\log(a_t) = \log(a_0) + \alpha t + \varepsilon_t$$

and found an estimate for α, for the timeframe 1951-94, at 0.0111, while Grasselli & Maheshwari (2017a) found 0.0155 for the period 1960-2010.

[6] The full derivation of the model is presented in the Appendix A.

[7] See Desai et al. (2006).

The labor force growth, β

The potential workforce, N, is assumed to grow exponentially at a coefficient $\beta > 0$:

$$\frac{\dot{N}}{N} = \beta.$$

Using a similar method for labor productivity, Harvie (2000) estimated this parameter, for the period 1951-94, at 0.0206, while Grasselli & Maheshwari (2017a) found 0.0165 for the period 1960-2010. .

The depreciation of capital, δ

As is standard, the stock of capital, K, is assumed to accumulate with respect to investment, I, and to depreciate at a constant rate, δ,

$$\dot{K} = I - \delta K.$$

Although in Harvie (2000) the depreciation rate of capital was not included in the model, Grasselli & Maheshwari (2017a), this parameter is assumed to be the mean value of the following timeseries

$$\delta^G := \frac{\text{Consumption of Fixed Capital in current prices}}{\text{Price deflator for gross fixed capital formation} \times \text{Net capital stock (2005)}}.$$

By doing so, they found a value for δ^G of 0.0521.[8] Using the above definition, δ depends on the level of the net capital stock, in particular on its initial value. In the database provided by AMECO, the level of capital is set using the rather strong assumption that the capital stock equals three times the nominal GDP in 1960 for every country. In other terms, to find the initial stock of capital, the methodology used by the AMECO is $K_{t_0=1960} = 3 \times GDP_{1960}$. Therefore, the level found for δ will change proportionally to the assumption regarding the initial capital.

An alternative methodology to compute the depreciation rate is provided by the Penn World Table 8.1 database (hereafter PWT8).[9] In PWT8, the investment is divided into six classes, with each class having its own depreciation rate.[10] Therefore, the aggregated depreciation rate of capital of the whole economy will depend on what the capital is made of, and by consequence, the depreciation rate of capital will be time-variant. Using PWT8, an approximation of the depreciation rate of capital can by made by taking the mean value. This value would be 0.0376 for US data from 1951 to 2011. Note that, in PWT8, the initial capital stock is based on the assumption of an initial capital-to-output ratio methodology. More precisely, an initial amount is assigned for each of the six classes.[11] As previously mentioned, the initial value of each of the six classes and the path taken by the investment will influence the path of the depreciation of capital.

[8] This computation is found using AMECO (the European Commission's annual macro-economic database from 1960 to 2010).

[9] Full details about the database are available in Feenstra et al. (2015)

[10] For the sake of clarity, structures (residential and non-residential) will have a depreciation rate of 2% while software will depreciate at 31.5% per year.

[11] The approach based on the steady state of the Solow model was considered and studied, but showed less stable results than linear regression techniques for a substantial number of countries. For further details, I refer to the Appendix C of Feenstra et al. (2015).

When using Bayesian techniques in dynamic stochastic general equilibrium modeling, the inference of the depreciation rate of capital–a structural parameter–suffers from a lack of identification and therefore cannot be estimated accurately. Therefore, δ is often assumed to be 0.025 per quarter, or put differently, roughly 10% on annual basis (for the Euro zone, see Smets & Wouters (2003); for the US, see Smets & Wouters (2007)).

No consensus emerges about the different methodologies used to find the accurate depreciation rate of capital. In such instances, for an annual frequency, one has three options: (i) 0.0521; (ii) 0.0376; or (iii) 0.10. Since each of these values leads to different behaviors of the Goodwin model–especially for the employment rate–taking one of them may have a strong influence on the behavior of the estimation. Hence, in the sequel, I will let the data speak during the estimation of this parameter without any prior assumption on the level of the depreciation rate of capital.

van der Ploeg model

van der Ploeg (1985) relaxes the assumption that capital and labor cannot be substituted by endowing the economy with a CES production function,

$$Y = C\left[bK^{-\eta} + (1-b)(\lambda^L L)^{-\eta}\right]^{-\frac{1}{\eta}},$$

where $C > 0$ is the factor productivity and $b \in (0;1)$ is the share of capital. The short-run elasticity of substitution between capital and labor is given by $\sigma := \frac{1}{1+\eta}$. It is worth recalling that the CES production function allows for three limit cases: (i) when $\eta \to +\infty$, one retrieves the Leontief production function; (ii) $\eta \to 0$ leads to the Cobb-Douglas production; (iii) if $\eta \to -1$ one recovers the linear production function.[12] Let us assume that the producer maximizes its profit given the wages .[13] It follows that the capital-to-output ratio is now endogenous. It is given by the first-order condition of profit maximization,

$$\nu(t) \quad := \quad \frac{K(t)}{Y(t)} = \frac{1}{C}\left(\frac{1-\omega(t)}{b}\right)^{-\frac{1}{\eta}}.$$

van der Ploeg (1985) illustrates the important structural instability property of Goodwin model. Indeed, a minor modification in the parameters of the Lotka-Volterra model can lead to radical change in the quantitative behavior of the economic model. For instance, a small perturbation on the elasticity of substitution ($\sigma \cong 0$), the phase-portrait changes from a center to a stable focus: the model with the CES production technology. The reduced two-dimensional system is,[14]

$$\begin{cases} \frac{d\omega_t}{\omega_t} &= \left(\frac{\eta}{\eta+1}\right)\left[\phi(\lambda_t) - \alpha\right] dt \\ \frac{d\lambda_t}{\lambda_t} &= \left(Cb^{-1/\eta}(1-\omega_t)^{1+1/\eta} - (\delta+\beta+\alpha)\right) dt - \frac{1}{\eta}\left(\frac{d\omega_t}{\omega_t(1-\omega_t)}\right) \end{cases} \tag{2}$$

[12] As in Goodwin's seminal version, wages are set conformly to the short run Phillips curve. On the other hand, we confine ourselves to a real economy, so that the consumption price is normalized to 1.

[13] This minimal rationality argument is analogous to the assumption in Goodwin's model that the allocation of capital and labor is always at the diagonal of the (K, L)-plan, so that we have not only $Y = \min\left(\frac{K}{\nu}, aL\right)$ but also $Y = \frac{K}{\nu} = aL$.

[14] The full derivation is available in Appendix A as a particular case. A slight change for simplicity and without any consequence has been made compared with van der Ploeg (1985) since the labor productivity is not taken into account in the wage bargaining process.

Since the assumptions of both models are similar, most of the parameters of the systems (1) and (2) look alike. The difference lies in the new parameters introduced by the CES production function. It also worth noting that one of the benefits of the van der Ploeg extension of the Goodwin model is that the trajectories taken by model (2) are less sensitive to small changes in the set of parameters than model (1).

Estimating those models parameter per parameter may lead to spurious results since certain key parameters for the dynamics, for instance δ, are very sensitive to the choice of the database and the methodology chosen to compute it. As a result, one can estimate the model as a whole with no identification assumptions for any of the parameters, especially for δ and ν. In what follows, the SMLE will be used to estimate the entire model. Before moving to this, however one needs to extend the model from deterministic to stochastic.

2.2 The stochastic extensions

In order to introduce stochastic perturbations in the system, some of the key assumptions of the Goodwin model are extended. These assumptions are on the wage dynamics (i.e. the growth of W) and the labor productivity dynamics (i.e. the growth of a).[15] Thus following assumptions will be used

- *Assumption 1:* The labor productivity is defined as,[16]

$$\frac{da_t}{a_t} := \alpha dt - \sigma_1(\cdot)dB_t^1,$$

  with B_t^1 a Brownian motion.

- *Assumption 2:* The real wages are set using a short-term stochastic Phillips curve,

$$\frac{dW_t}{W_t} := \Phi(\lambda_t)dt + \sigma_2(\cdot)dB_t^2,$$

  with B_t^2 a Brownian motion independent from B_t^1.

The first assumption is borrowed from Nguyen-Huu & Costa-Lima (2014) while the second has a twofold motivation: (i) the short term Phillips curve has been noisy throughout the second half of the last century;[17] and (ii) the estimation procedure requires to have a invertible covariance matrix permitted by this extra assumption.[18] When applying both assumptions to the model (1), one retrieves the following model[19]

$$\begin{cases} \frac{d\omega_t}{\omega_t} &= \left(\Phi(\lambda_t) - \alpha + \sigma_1(\cdot)^2\right) dt + \sigma_1(\cdot)dB_t^1 + \sigma_2(\cdot)dB_t^2 \\ \frac{d\lambda_t}{\lambda_t} &= \left[\frac{(1-\omega_t)}{\nu} - (\alpha + \beta + \delta) + \sigma_1(\cdot)^2\right] dt + \sigma_1(\cdot)dB_t^1. \end{cases} \tag{3}$$

It is important to note that if $\sigma_2 = 0$ (i.e. whenever the short-term Phillips curve is deterministic), then one recovers the model of Nguyen-Huu & Costa-Lima (2014). Following

[15] See Appendix A for more details on the model's derivation.

[16] This assumption is slightly modified for the van der Ploeg extension of the Goodwin model. See Appendix ?? for further details.

[17] See Blanchard (2016b) for further details.

[18] More detailed shortly.

[19] See Appendix A for the full derivation of the stochastic models.

the insights of Nguyen-Huu & Costa-Lima (2014) on the dynamics behavior within $\mathcal{D}$, I will apply the estimation methodology under the following additional assumptions on system (3)

- *Assumption 3:* $\Phi : \lambda \in [0,1) \longmapsto \Phi(\lambda) = \frac{\phi_1}{(1-\lambda)^2} + \phi_0$
- *Assumption 4:* $\sigma_1(\cdot) = \sigma_1(1-\lambda)^{1/1000}$
- *Assumption 5:* $\sigma_2(\cdot) = \sigma_2(1-\lambda)^{1/1000}(\lambda)^{1/1000}$.

Note that, when applying assumptions 1,2,3,4, and 5 on the boundaries of the domain $\mathcal{D}$, that is

$$\Big(\mathcal{B}((\omega_0,\varepsilon_1),\varepsilon_2)\bigcup\mathcal{B}((\varepsilon_3,\lambda_0),\varepsilon_4)\Big)\bigcap\mathcal{D}$$

with $\mathcal{B}$ an open ball in $\mathbb{R}^2$, $\varepsilon_i > 0\ \forall i = 1,\ldots,4$ and $(\omega_0,\lambda_0) \in \mathbb{R}^2$, system (3) dynamics is closely akin to the one of Nguyen-Huu & Costa-Lima (2014). It is, therefore, very likely that this system is staying within the domain $\mathcal{D}$ in accordance with Nguyen-Huu & Costa-Lima (2014).[20]

When applying these assumptions to the model (2), we get

$$\begin{cases} \frac{d\omega_t}{\omega_t} &= \left(\frac{\eta}{\eta+1}\right)\Big\{\phi(\lambda_t) - \alpha - \frac{1}{2}\left(\frac{1-\eta}{(1+\eta)^2}(\sigma_1(\cdot)^2+\sigma_2(\cdot)^2) - \frac{\sigma_1(\cdot)^2}{\eta+1}\right)\frac{\sigma_2(\cdot)^2}{\eta+1} \\ &\quad + \left(\frac{\sigma_1(\cdot)\eta}{1+\eta}\right)^2 + \left(\frac{\sigma_2(\cdot)}{1+\eta}\right)^2\Big\}dt + \left(\frac{\eta}{\eta+1}\right)\sigma_1(\cdot)dB_t^1 + \left(\frac{\eta}{\eta+1}\right)\sigma_2(\cdot)dB_t^2 \\ \frac{d\lambda_t}{\lambda_t} &= \left(Cb^{-1/\eta}(1-\omega_t)^{1+1/\eta} - (\delta+\beta+\alpha)\right)dt \\ &\quad - \left(\frac{\omega_t}{1-\omega_t}\right)^2\left(\frac{1}{1+\eta}\right)\left(\frac{\sigma_1(\cdot)^2+\sigma_2(\cdot)^2}{2}\right)dt - \left(\frac{1-\eta}{(1+\eta)^2}\frac{(\sigma_1(\cdot)^2+\sigma_2(\cdot)^2)}{2} - \frac{\sigma_1(\cdot)^2}{\eta+1}\right)dt \\ &\quad + \left(\frac{\omega_t}{1-\omega_t}\right)\left\{-\eta\left(\frac{\sigma_1(\cdot)}{1+\eta}\right)^2 + \left(\frac{\sigma_2(\cdot)}{1+\eta}\right)^2\right\}dt + \left(\frac{\eta}{\eta+1}\right)^2\sigma(\cdot)_1^2dt + \left(\frac{\sigma_2(\cdot)}{1+\eta}\right)^2dt \\ &\quad + \left[\left(\frac{\omega_t}{1-\omega_t}\frac{1}{1+\eta}\right)\right]^2(\sigma(\cdot)_1^2+\sigma_2(\cdot)^2)dt - \frac{1}{\eta}\left(\frac{d\omega_t}{\omega_t(1-\omega_t)}\right) + \sigma_1(\cdot)dB_t^1 \end{cases} \tag{4}$$

It is worth mentioning that, if $\eta \to +\infty$, and if $A = 1/\nu$, model (4) boils down to model (3), and if in addition $\sigma_1 = \sigma_2 = 0$, those models are similar to the deterministic case, 1. Finally, putting together both the rationale through which system (3) stays within the domain $\mathcal{D}$ and the inward pointing property of the van der Ploeg extension of the Goodwin model, make very likely that system (4) to have an existing solution that stays in $\mathcal{D}$.

3 The estimation technique

This section aims to present the methodology used for the estimation and address the identification issues.

In previous attempts (Harvie (2000), Grasselli & Maheshwari (2017a)) among others), the Goodwin model was estimated equation by equation.[21] Each parameter was estimated separately using standard econometric tools such as an OLS, an error correction model or a vector error correction model. Mixed conclusions were drawn from those studies: in particular the long run equilibrium found was hardly consistent with phase space (ω,λ) shown by the data. In order to find better results, Harvie (2000)

[20] The proof is left for further research.

[21] As pointed out by Blanchard (2016a), equation by equation estimation procedure can be at odds with the actual model dynamics.

pointed out that certain theoretical extensions of the Goodwin model such as Desai (1973), aid in aligning the cyclical behavior given by the data. However, he also pointed out that the Goodwin model is econometrically challenging to estimate, and that additional extensions make the model more difficult to estimate empirically.

Instead of proposing the estimation of new theoretical extensions of the Goodwin model, this Section aims at providing another estimation approach for such models. Rather than estimating the model parameter by parameter, I directly estimate the whole nonlinear dynamical system. The most obvious benefit of the approach is that the estimation of the depreciation rate of capital, δ, and hence the whole model, does not rely on the assumption of the level of the initial capital stock made by the database under consideration as previously mentioned. The estimation will instead be based on the estimation of multidimensional SDEs.

3.1 Sketch of the SMLE

SDEs are wildly used in finance, pricing theory (see Black & Scholes (1973)), yield curve models (see the HJM model from Heath et al. (1990)), and algorithmic trading among others. Financial markets, rather than macroeconomics, are more suitable for a SDEs setting because data is often available at a high frequency.[22] Nonetheless, tools to infer the ability of SDEs to cope with lower frequency data have been developed.

The estimation methodology is borrowed from Durham & Gallant (2002) and is extended to the multivariate framework.[23] Let us consider a reduced-form SDE on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$\begin{cases} dX_t &= f(X_t)dt + g(X_t)dB_t, \\ X_{t_0} &= X_0. \end{cases}$$

Where $X_t \in \mathbb{R}^n$ is the state variable vector, B_t is a d-dimensional Brownian motion, $f : \mathbb{R}^n \to \mathbb{R}^n$ is the drift of the process and $g : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ is the diffusion. For the sake of clarity, $X_t = (\omega_t, \lambda_t)^T$, where T is the transpose operator.

Ideally, to compute the maximum likelihood estimation, one should know the transition density. Because analytic solutions are rarely available in practical situations, the transition densities must be approximated numerically. Therefore, numerical methods are required to approximate their solutions. In what follows, the Euler-Maruyama scheme is used (see Kloeden & Platen (1992)). On the one hand, the Euler-Maruyama is computationally intensive in minimizing the error of the numerical methods, but on the other hand, this scheme is computationally feasible at all times in multivariate framework. For instance, if one uses the scheme proposed by Jimenez et al. (1999), one should keep in mind the authors' caution:

> [...] this numerical scheme is not always computational feasible since it can fail for SDE for which the Jacobian matrix $J_f^{-1}(X)$ is singular or ill-conditioned in at least a point. (Jimenez et al. (1999), p.593)

For the sake of clarity, the Euler-Maruyama scheme is

$$\tilde{X}_{i+1} = \tilde{X}_i + f(\tilde{X}_i)\delta + g(\tilde{X}_i)\delta^{1/2}\varepsilon_i$$

[22] The standard time mesh can be some fractions of a second. Such high-frequency data is not available in macroeconomics where the time mesh is often a quarter, or perhaps a year.

[23] What follows is a sketch of the methodology, an extensive explanation is available in Appendix B.

where $\delta = t_{i+1} - t_i$, $\varepsilon_i \sim \mathcal{N}(0,1)$, and $\tilde{X}$ is the approximated counterpart of X. Under some mild assumptions, it can be shown that this approximation converges to the true maximum likelihood. Nevertheless, the approximation may not be sufficiently accurate for the sampling frequencies, especially for macroeconomic data. The general idea of the SMLE method is to obtain the true transition probability, $p(x_t, t; x_s, s)$. Using the Euler-Maruyama scheme, one can approximate the true transition density by $p^{(1)}(x_t, t; x_s, s)$. As previously mentioned, the frequency is too low to provide a good convergence to the true maximum likelihood estimation. An idea is to generate a subinterval $s = \tau_1 < \ldots < \tau_M = t$, so that the random variable is sufficiently accurate at each subinterval. The vector $(X(\tau_2), \ldots, X(\tau_{M-1}))$ is therefore unobserved and should be simulated by a Brownian bridge. Because the process is Markovian, one obtains

$$\begin{aligned} p(x_t, t; x_s, s) &\approx p^{(M)}(x_y, t; x_s, s) \\ &:= \int \prod_{m=0}^{M-1} p^{(1)}(u_{m+1}, \tau_{m+1}; u_m, \tau_m) \\ &\quad \times d\lambda^{Leb}(u_1, \ldots, u_{M-1}) \end{aligned}$$

where λ^{Leb} is the Lebesgue measure. Each subpaths are simulated using the Girsanov theorem and, therefore, an important sampler.[24] The integral can be evaluated using Monte Carlo integration. By doing so, one obtains the simulated transition probability, $p^{(M,K)}(x_y, t; x_s, s)$, where K is the Monte-Carlo parameter. By repeating this operation for each transition of the dataset, I compute the simulated likelihood.[25]

3.2 Identification issues

To infer the model (3) with the Leontief production function, it is necessary to order the model from the parameters that cannot be identified. Indeed, in model (3) the estimation procedure does not distinguish between β and δ. In the following, I suggest two counterparts that will be estimated:

$$\begin{cases} \frac{d\omega_t}{\omega_t} &= (\Phi^*(\lambda_t) - \phi_0 + \sigma_1^2)\, dt + \sigma_1 dB_t^1 + \sigma_2 dB_t^2 \\ \frac{d\lambda_t}{\lambda_t} &= \left[\frac{(1-\omega_t)}{\psi_0} - \psi_1 - \sigma_1^2\right] dt + \sigma_1 dB_t^1 \end{cases}$$

where $\Phi^*(\lambda_t)$ is the short term Phillips curve without constant; and ϕ_0 is the constant of the short term Phillips curve minus the labor productivity. ψ_0 remains the capital-to-output ratio, while ψ_1 is the combined parameter of $(\alpha+\beta+\delta)$. Turning to the model (4), the CES production function, the same specifications for ϕ_0 and ψ_1 are made. The only difference is that for $Cb^{-1/\eta}$, the idiosyncratic effect of C and b cannot be distinguish; for the estimation it will be denoted by C_b. It is worth mentioning that parameter η, which controls the substitution between capital and labor, is well defined since it will

[24] See Appendix B for an extensive discussion of an important sampler adapted for systems (3) and (4) to stay in the $\mathcal{D}$ domain.

[25] Throughout the paper, the mean value of distribution of the likelihood will be reported.

weight the influence of the wage share dynamics on the employment rate dynamics.

$$\begin{cases} \frac{d\omega_t}{\omega_t} & = \left(\frac{\eta}{\eta+1}\right)\left\{\Phi^*(\lambda_t) - \phi_0 - \frac{1}{2}\left(\frac{1-\eta}{(1+\eta)^2}(\sigma_1^2+\sigma_2^2) - \frac{\sigma_1^2}{\eta+1}\right)\frac{\sigma_2^2}{\eta+1} \right. \\ & \left. + \left(\frac{\sigma_1\eta}{1+\eta}\right)^2 + \left(\frac{\sigma_2}{1+\eta}\right)^2\right\} dt + \left(\frac{\eta}{\eta+1}\right)\sigma_1 dB_t^1 + \left(\frac{\eta}{\eta+1}\right)\sigma_2 dB_t^2 \\ \frac{d\lambda_t}{\lambda_t} & = \left(C_b(1-\omega_t)^{1+1/\eta} - \psi_1\right) dt \\ & - \left(\frac{\omega_t}{1-\omega_t}\right)^2 \left(\frac{1}{1+\eta}\right)\left(\frac{\sigma_1^2+\sigma_2^2}{2}\right) dt - \left(\frac{1-\eta}{(1+\eta)^2}\frac{(\sigma_1^2+\sigma_2^2)}{2} - \frac{\sigma_1^2}{\eta+1}\right) dt \\ & + \left(\frac{\omega_t}{1-\omega_t}\right)\left\{-\eta\left(\frac{\sigma_1}{1+\eta}\right)^2 + \left(\frac{\sigma_2}{1+\eta}\right)^2\right\} dt + \left(\frac{\eta}{\eta+1}\right)^2 \sigma_1^2 dt + \left(\frac{\sigma_2}{1+\eta}\right)^2 dt \\ & + \left[\left(\frac{\omega_t}{1-\omega_t}\frac{1}{1+\eta}\right)\right]^2 (\sigma_1^2+\sigma_2^2) dt - \frac{1}{\eta}\left(\frac{d\omega_t}{\omega_t(1-\omega_t)}\right) + \sigma_1 dB_t^1 \end{cases}$$

4 Data and the Estimation Results

This section presents the data sources and methodology to construct the phase variables (ω, λ) are discussed. Secondly, data's properties are examined. Finally, different specifications for the short term Phillips curve are derived.

4.1 Data Construction and Preliminary Analysis

Data used for the estimation are taken from two main sources: (i) U.S. Bureau of Economic Analysis; and (ii) U.S. Bureau of Labor Statistics. The frequency of the data is quarterly and runs from 1948:Q1 to 2017:Q2. The two main variables used are the labor share, ω, and the employment rate, λ.The employment rate is defined as:[26]

$$\lambda := \frac{\text{Total Employment}}{\text{Total Labor Force}}.$$

The wage share is

$$\omega = \frac{\left(1 + \frac{\text{Self Employed}}{\text{Total Employees}}\right)\text{CE}}{\text{GDP at factor cost}},$$

where CE stands for the compensation of employees, which is the total gross (pre-tax) wage paid by employers to employees within a single quarter. Although a large part of the total wages earned in the economy is determined by the compensation of employees, a substantial amount is located in the gross operating surplus (hereafter GOS) due to the self-employed (while it represented more than 18% of the total workers in 2015, this category dropped down to 8% in 2015).[27] In order to have a more realistic measure of the weight of the wage in the economy, one can make the assumption that the self-employed, on average, earn as much as employees. Taking this assumption, I add a proportional share, representing the wages earned by self-employed, to the CE (Self Employed/Total Employees).[28] Turning to the denominator of ω, GDP is measured at factor cost. Since the income approach of the GDP at market price is

$$\begin{aligned} \text{GDP (market price)} &= \text{CE} + \text{GOS} + \text{T-S} \\ \text{GDP (market price)} - \text{T-S} &= \text{CE} + \text{GOS} \\ &= \text{GDP (factor cost)} \end{aligned}$$

[26]The definition of the labor share is similar to Harvie (2000).
[27]This idea is discussed extensively in Mohun & Veneziani (2006).
[28]This methodology is borrowed from Grasselli & Maheshwari (2017a).

where GOS can be read as the EBITDA (earnings before interests, taxes, depreciation, and amortization), and T-S is the net taxes on products and imports.[29] Figure 1 repre-

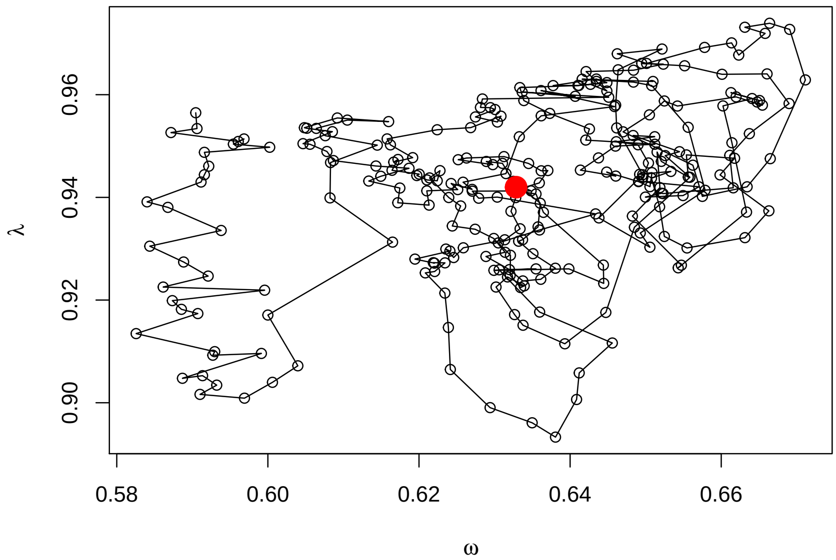

Figure 1: The empirical phase portrait of the variable (ω, λ). In red, the empirical mean of the state variables.

sents the empirical phase portrait of the state variables ω, on the x-axis and λ, on the y-axis. Using qualitative evaluation on similar data sets, Solow (1990), Harvie (2000) and Mohun & Veneziani (2006), showed that the data have a clockwise behavior, as would be expected from Goodwin's theory. It is worth noting that in the left part of the quadrant, the last cycle, that started in 2007:Q4, is the most at odds with previous cycles. This inconsistency is mainly due to current wage shares being lower than those that were explored over the sample. Also, one can note, from a qualitative perspective, SDEs should provide a feasible modeling if one wants to replicate such kind of trajectory. The red dot in Figure 1 represents the $(x, y)-$coordinate of the empirical means of the phase space. Despite that this red dot is, qualitatively, at the center of the portrait, it seems that multiple cycles are represented in Figure 1, subsection 4.2 shows some evidences with descriptive statistics.

4.2 First evidences of structural changes

There is a large body of recent macroeconomic literature that focuses on changes in the relationship (causality, dependency, explanatory strength, etc.) between macroeconomic variables such as GDP, oil price, consumption, investment among others. Those changes are referred as structural breaks (or regime switching). Various methodologies could be considered, for instance, Kim & Nelson (1999) and Perez-Quiros & McConnell (2000) have used a volatility reduction Markov switching model and independently found a structural break at the date 1984:Q1. Later, using an alternative approach, Stock & Watson (2003) confirms that the volatility of macroeconomic variables has declined at the aftermath of the FED's aggressive response to inflation, during the Volcker era, that was credited to end the United States' stagflation crisis of the 1970s. Figure

[29] For the sake of clarity, it can be seen as the V.A.T., or subsidies such as environmental externalities.

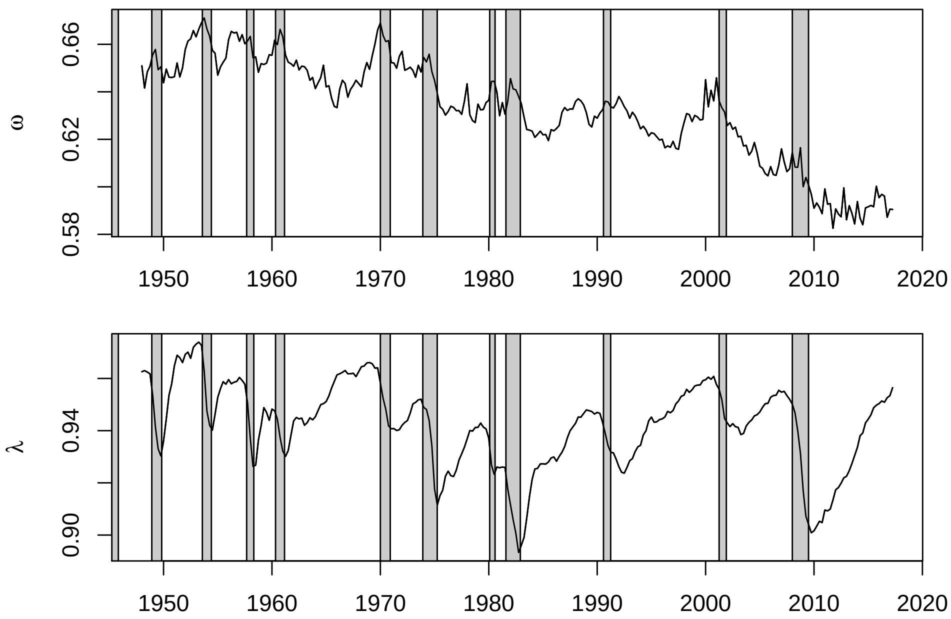

Figure 2: Timeseries of the wage share (top) and the employment rate (bottom). The shaded grey represents NBER recessions.

2 shows the evolution of the wage share (top) and the employment rate (bottom) over time. The shaded areas refer to the NBER recession periods. Those recession periods are highlighted since they somehow represent the end of the Lotka-Voletrra cycle symbolized by the drop in the employment rate and, hence, may be the premise of a new cycle era. While the wage share qualitatively shows a downward bending long term trend that plateaued over the last five to ten years, interestingly, the employment rate shows first a decreasing trend until mid-1980s' and, then, an upward trend until the subprime mortgage crisis (the last grey-shaded area). For the sake of descriptive

Sub-periods	Mean of ω	St.dev. of ω	Mean of λ	St. dev. of λ
1948:Q1 - 1984:Q1	0.648	0.011	0.945	0.017
1984:Q2 - 2000:Q1	0.627	0.007	0.940	0.010
2000:Q2 - 2017:Q2	0.606	0.016	0.938	0.018

Table 1: First and second empirical moment of (ω, λ) for given sub-periods.

statistics, table 1 presents the empirical mean values and the standard deviations of the state variables over different time frame.[30] The results show the decline of volatility, as documented in Stock & Watson (2003), over the first (1948:Q1-1984:Q1) and the second (1984:Q2-2000:Q1) sub-period and also the downward shift of the mean of the state variable ω that loses almost two standard deviations from the first to the second sub-period. On the other hand, the last sub-period shows that, on average, wage-to-GDP ratio and the employment rate levels are lower than previous sub-periods with a return to a relatively high volatility era equivalent to sub-period one. As illustrated by Figure 3, the observations around the empirical mean values, represented by a red dot, of each sub-period show less dispersion that in Figure 1. Therefore, qualitatively,

[30]The motivation behind the selected time frame will be discussed shortly.

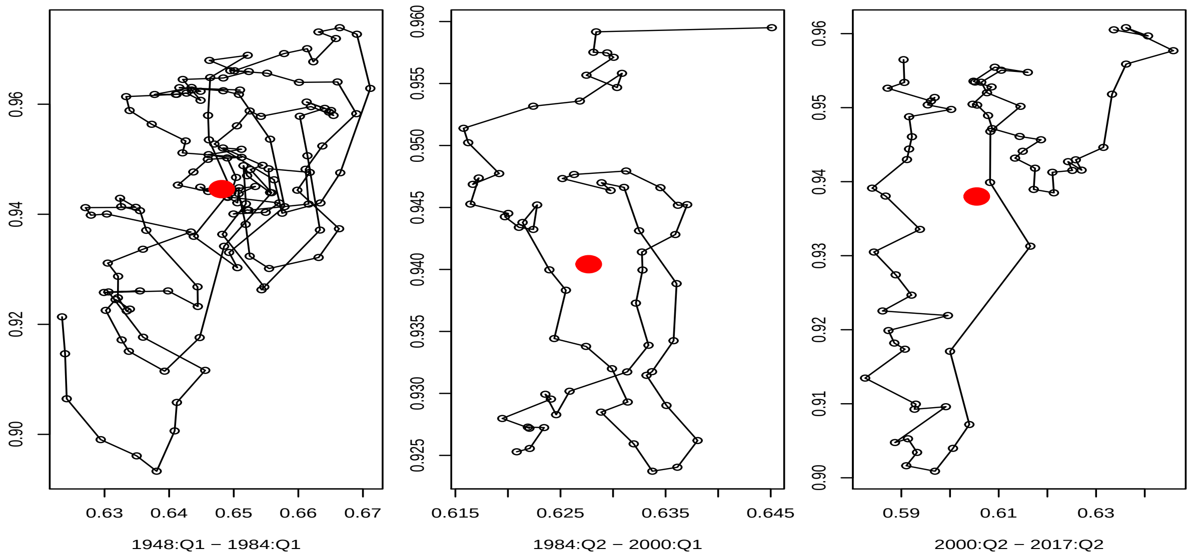

Figure 3: Phase space of the three sub-periods.

and as shown in Mohun & Veneziani (2006), one can conclude that various cycles with different frequencies and equilibrium can be found in the data.

For the sake of completeness, Figure 4 shows each sub-period on the same scale with different colors: (i) black for the period 1948:Q1-1984:Q1; (ii) the period 1984:Q2-2000Q1 is represented with the color blue; and (iii) red illustrates the last period 2000:Q2-2017:Q2. Dots represent the empirical mean coordinate of their respective colors. The downward sloping trend of the wage share over time in the empirical observation is well illustrated in this graph since, along the sub-periods, the scatter plots, as well as the empirical means, are heading from the north west towards the south west of the phase diagram.

4.3 Short term Phillips curve

A degree of freedom for the global behavior of the dynamic and allowed by Goodwin (1967) lies in the short term Phillips cure. Perhaps, in Goodwin's estimation framework, previous attempts to estimate the phenomenological behavioral function were essentially made using OLS, see Harvie (2000), with the aim of estimating:

$$\frac{\dot{w}}{w} = \phi(\lambda). \tag{5}$$

Nevertheless, using such a framework to estimate differential equation such as the Phillips curve may lead to spurious results. For simplicity, consider that the time between t and $t+1$ is one year, w_t is the real wage, and λ_t is the employment rate. Additionally, suppose that using quarterly data, the following linear regression is well specified (meaning that the residuals pass standard tests)

$$\log\left(\frac{w_{t+1/4}}{w_t}\right) = \alpha_0 + \alpha_1\lambda_t + \varepsilon_{t+1/4}.$$

Taking the deterministic part, one can rewrite the same equation as

$$\int_t^{t+1/4} \frac{dw_t}{w_t} = \alpha_0 + \alpha_1\lambda_t,$$

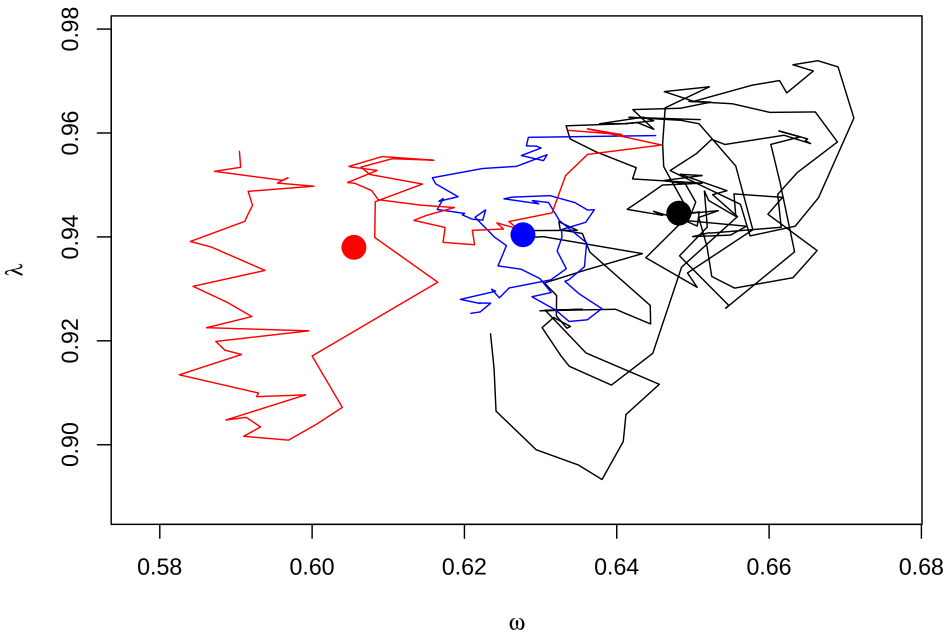

Figure 4: Phase space of the three sub-periods.

and by taking the first derivative with respect to t, we are led to a differential equation with delay:

$$\frac{\dot{w}_{t+1/4}}{w_{t+1/4}} = \frac{\dot{w}_t}{w_t} + \alpha_1 \dot{\lambda}_t,$$

which involves the theory of delay differential equations. Therefore, if one wants to infer the short term continuous Phillips curve using OLS, one does not obtain the desired equation 5 because of the discretization bias.[31]

The estimation methodology proposed in this paper allows the inference of the short term Phillips curve from its original specification in the context of the Goodwin model. As stated by assumption 3, this paper will focus on the following specification:

$$\phi(\lambda) = \phi_0 + \frac{\phi_2}{(1-\lambda)^2}. \tag{6}$$

4.4 Estimation results

The aim of this section is twofold: (i) to present the fitting of the estimation of the whole model, this has to be understood as a proof of concept for the SMLE approach; and (ii) to test for structural breaks in the cycle.

4.5 The fitting

As a preliminary exercise, this section presents the estimation of models (3) and (4) over the whole sample (1948:Q1-2017:Q2). The quality of the fitting will be measured by the well-known AIC criterion (see. Akaike (1973)). It is defined as

$$\text{AIC} := -2 \times \log(\text{Likelihood}) + 2 \times \text{Number of parameters}.$$

[31] I refer the reader to Appendix D for the test on some data generating process.

This measure allows for the assessment of the relative quality of statistical models for given datasets. This model selection procedure results in a trade off between the goodness of fit–the log-likelihood–and the number of estimated parameters. The model that has the minimal value for the AIC criterion would be qualitatively the best to fit the data.In the following, each likelihood is computed using first $M = 8$ and $K = 8$ and second with $M = 16$ and $K = 128$. This two-stage procedure enables to reach the optimum faster. According to the AIC criterion on the full sample, the result provided

Leontief (3)	CES (4)
-5417.29	-4391.25

Table 2: The AIC values of the Leontief and the CES models.

by table 2 is that the modern version of the Goodwin model is a significantly better candidate to explain the data's behavior.

4.6 The parameter estimates

In aiming to test the economic reliability, over the whole sample, of the parametric estimation strategy, this section displays the estimates found for model (3), with a Leontief production function and model (4), the CES production function counterpart. Results will then be discussed.

4.6.1 The Leontief production function

ϕ_1	ϕ_0	ψ_0	ψ_1	σ_1	σ_2
$1.902e-05$	0.0084	6.564	0.056	0.0077	0.01589

Table 3: The parameter estimates of model (3).

Interestingly, the capital-to-output constant ratio, ψ_0, is estimated to be 6.6. This value is at odds with previous findings in the sense that it appears overestimated. On the other hand, ψ_1 displays a value of about 5.6%. Since it represents the compounded value of $(\alpha + \beta + \delta)$, and by taking reasonable values for α and β previously discussed, this would mean that δ is inferred to be approximately 2.5%, lower than what as been previously discussed. As previously mentioned, this value is the result of the estimation on the whole sample. However, in the case of structural change (or nonlinearity) in the timeserie, this estimate may lead to spurious results as it may be in this case here. This problem will be addressed shortly. One can note that the parameter estimates for each of the short term Phillips curves are positive and significant.

4.6.2 The CES Production Function

ϕ_1	ϕ_0	C_b	ψ_1	η	σ_1	σ_2
$2.181e-05$	0.0095	0.1296	0.0412	6.2913	0.0104	0.0130

Table 4: The parameters estimate of model (4).

The estimated parameters revealed in table 4 show that the ϕ_0 and ϕ_1, namely the short term Phillips curve parameters, are of the same magnitude to the previous estimates. The remaining parameters $(C_b, \psi_1, \eta, \sigma_1, \sigma_2)$ are relatively close to each other. The major difference is in the elasticity of substitution between capital and labor; it is approximately equal to 0.1371 $(\approx 1/(\eta+1))$.

4.7 Structural Breaks

Tests for structural breaks have been extensively studied in timeseries analysis. Back in 1960, to the best of my knowledge, Chow (1960) published the first paper that tests parameter change in a linear regression. More recently, Bai & Perron (2003) showed more advance techniques for multiple breaks in the timeseries. Here, with the SMLE estimation framework, in order to detect a structural break, I minimize the following criterion:

$$\text{BIC} := -2\log(\text{Likelihood}) + \log(\text{Sample Size}) \times \text{Number of parameters}.$$

This model selection criterion was first introduced by Schwarz (1978) and stands for Bayesian information criterion, or BIC. Compared to the AIC, the BIC penalizes more the number of parameters by putting more weight, hence it prevents for overfitting. This choice is motivated by the fact that a structural break will increase the number of parameters, by making them time-varying. Thus, it will be harder to detect a break using this technique, unless it significantly improves the likelihood of the estimation on the whole sample. Table 5 presents the location where the BIC and AIC criteria is

Number of structural break(s)	0	1	2
BIC Leon	**-5395.5**	-5384.5	-5372
AIC Leon	-5417.3	-5428	**-5437.3**
Located break(s)	x	2006:Q3	1984:Q1 and 1999:Q2
BIC CES	-4365.9	-4466.6	**-4475.4**
AIC CES	-4391.2	-4517.3	**-4551.5**
Located break(s)	x	2010:Q3	1983:Q2 and 1999:Q3

Table 5: BIC and AIC criteria of model 3 for 0, 1 and 2 structural breaks and their location.

minimal for zero, one, and two structural breaks of both models (3) and (4). Unexpectedly, the size of the penalty for a given criterion greatly matters when analyzing the results. Indeed, if one relies on the BIC, the model that has the better accuracy in explaining the data would be model (3) without break. However, if one considers the AIC model (3) with two breaks would be considered. The conclusion according to the structural breaks is therefore mixed and the remaining of the paper will consider both cases (without and with two structural breaks). However, the one structural break case undoubtedly shows the weakest results. Moreover, table 5 shows that for every specification model (3) uniformly outperforms model (4). Interestingly, the breaks are located approximately in 1984:Q1, which is consistent to the break date discussed in Subsection 4.2 and in the year 1999 that represents a change that occurred slightly before the dot-com bubble crises.

	ϕ_1	ϕ_0	ψ_0	ψ_1	σ_1	σ_2
1948:Q1 - 1984:Q1	$1.548e-5$	0.0086	1.71	0.201	0.00914	0.0163
1984:Q2 - 1999:Q2	$1.475e-5$	0.0047	2.9516	0.124	0.00364	0.0087
1999:Q3 - 2017:Q2	$6.55e-6$	0.0042	3.1013	0.126	0.00633	0.0197

Table 6: The parameter estimates of model (3) over the different sub-periods.

Estimated parameters of model (3) are displayed in table (6). Over the different sub-periods, three main differences with table (3) are worth mentioning. First, the short term Phillips curve parameters shows a lot of variability throughout the periods. This finding is not new, the short term Phillips cure has shown a different pattern in the second half of the past century (see Blanchard (2016b)). A conclusion would be that the nonlinearity for the short term Phillips curve could have changed in the last sub-period. Second, the values found for the capital-to-output ratio, ψ_0, are almost in line with the literature for each attempt. Remember, without any prior assumption on the

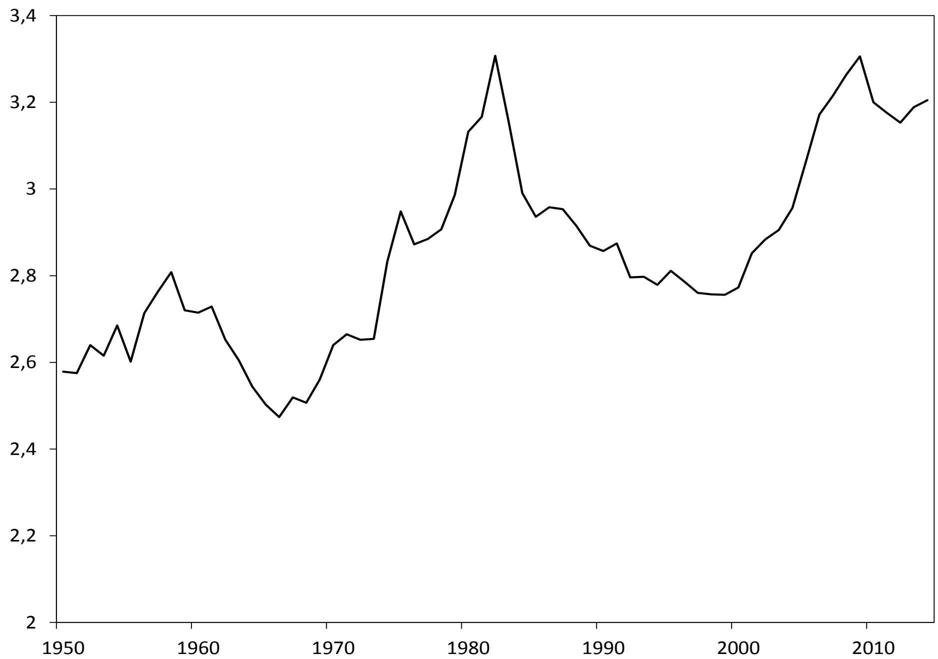

Figure 5: The capital-to-output ratio of the United-States - 1950-2013. Source: PWT8.

level of the capital-to-output ratio, the previous finding was approximately 6.6. Figure 5 shows the variation of the ratio over the period 1950-2013 according to the PWT8 database. Within sub-period 1984:Q2-1999:Q2, this ratio oscillates around 2.8, while the value found is 2.95 and for the sub-period 2000:Q2-2015:Q4 the latter fluctuates around 3, close to the estimated value 3.1. The change in that estimate generates an adjustment of ψ_1, in line with the literature, compared with the estimates of the model with no break. This value increases from 6% to approximately 13% that implies a depreciation rate of capital of about 10% per year. Third, parameters σ_1 and σ_2, namely the volatility of the data, reflect the great moderation era by showing a lower volatility for the second sub-period with respect to the others. Table 7 presents the estimated parameters for model (4). It shows similar conclusions for the estimates of the short term Phillips than for the Leontief counterpart for the first sub-period and then diverge.

Dates	ϕ_1	ϕ_0	C_b	ψ_1	η	σ_1	σ_2
48:Q1-83:Q2	$1.507e-5$	0.0081	0.480	0.136	4.402	0.0128	0.0127
83:Q3-99:Q3	$2.489e-5$	0.0069	0.446	0.140	6.51	0.0049	0.0074
99:Q4-17:Q2	$-2.020e-6$	0.0046	0.337	0.130	30.2	0.0693	0.0184

Table 7: The parameters estimate of model (4).

One can note that the estimates of ϕ_0 are similar for the first and second sub-periods. Also, the η parameters keep increasing along the sub-periods meaning that capital-labor substitution keeps decreasing along the time, estimates of the elasticity of substitution are chronologically 18.51%, 13.32%, and 3.21%. Likewise the previous estimates of the Leontief production technology, the estimates of the volatility parameters present similar conclusion in the CES case.

5 The backtesting

This section shows the methodology that is used to implement the backtesting. Moreover, the results for the no break case and the one with two breaks are displayed and analyzed.

5.1 The Methodology

In seeking to measure the performance of the model, the proposed methodology was inspired by Kilian & Vigfusson (2013) among others. The underlying concept is based on out-of-sample error forecast. The structure for the backtesting strategy is outlined in the following: Suppose that one has a dataset starting at T_0 and ending at T_1,

- Step 1: Choose a date between T_0 to T_1, say T^*.
- Step 2: Make an estimation of the model from T_0 to T^*.
- Step 3: Taking the deterministic form of the model into consideration, run a simulation for h-periods.
- Step 4: Compare, using distance d, the simulated value to the realized value obtained in Step 3.
- Step 5: Repeat steps 2-4 by increasing the T^* by one period through the end of the sample.

To measure the performance of the forecast, assume the following distance:

$$d_\gamma(x, y) = |x - y|^\gamma.$$

The h-period-ahead forecast performance will be evaluated in the following way: by considering m periods,

$$D_h^M = \sum_{j=1}^{m} d_\gamma(x_{i_h}^M, x_{i_h}^t),$$

where $x_{i_h}^M$ is the h-period-ahead forecast of the model M estimated from the beginning of the sample to the i-th period and, $x_{i_h}^t$ is the realized value at date $i + h$. In the following exercise, we will estimate the model in two ways: First, using the full sample; and Second, using a subsample of 150 (or 37.5 years) points,[32] $\gamma = 1$,[33] $h = 1, \ldots, 8$, and $m \approx 180$.[34] To test the relative performance of the forecast, a VAR will be used as the benchmark model. Using the AIC criterion, the optimal lag for the benchmark VAR model is 3.

5.2 The Results for models with no break

	Leontief		CES	
	ω	λ	ω	λ
h = 1	**0.12003**	**0.11825**	**0.11866**	**0.12077**
h = 2	**0.16428**	**0.21614**	**0.15969**	**0.22662**
h = 3	**0.20616**	**0.33573**	**0.20320**	**0.34248**
h = 4	**0.23916**	**0.47210**	**0.23752**	**0.47483**
h = 5	**0.28547**	**0.61642**	**0.28346**	**0.61812**
h = 6	**0.33063**	**0.77641**	**0.31757**	**0.78145**
h = 7	**0.37227**	**0.93727**	**0.34162**	**0.94490**
h = 8	**0.41832**	1.08931	**0.38212**	1.07286

Table 8: Relative performance of the model against a VAR.

Table 8 shows the forecasting performance of the Goodwin models under consideration against a VAR. Values below unity, in bold, indicates that the underlying model is globally performing better than a VAR for the state variable named by the column. For instance, 0.12003 means that the 1-period-ahead forecast of the Goodwin model endowed with a Leontief production function performs roughly 88% better than a VAR model for the state variable ω.

The results in table 8 are very promising. Both phase variables, ω and λ, are forecasted with far better accuracy with the SDEs presented in this paper than with the VAR process, especially in the short run. The relative performance decreases throughout the quarters ahead although it still shows very good performance. It is worth mentioning that the state variable λ shows sudden decrease in accuracy to the extent that for the 8^{th} quarter-ahead forecast the VAR would do better. Finally, no clear difference can be noticed between Leontief and CES technologies.

[32] A large evaluation period as been chosen in order to avoid any bias on the choice of the sample that can change the resulting outcome. The sensitivity analysis is available upon request.

[33] Similar results are found after a sensitivity analysis with $\gamma = 2$.

[34] For the forecast, one can use the simulation made by taking the mean of hundreds of simulated paths of the stochastic differential model. For computational purposes, 25 trajectories are made from the stochastic model under consideration, the median value is used as the forecast value.

5.3 Results for models using a rolling window estimation

Results of the previous exercise is made using the whole sample, however, as previously discussed, estimated parameters may be biased due to the structural breaks.[35] Therefore, for the sake of comparison, I perform the same exercise by estimating the model on a rolling window. In using this technique, the length of the dataset remains throughout each estimation. Meaning that, the methodology presented *supra* does not hold up anymore because at each iteration of the loop, the dataset size was strictly increasing. In this exercise, I chose a sample size of 150 (or 37.5 years),[36] this window size will stay fixed along the backtesting, meaning that as soon as a new forward looking data point is taken, the first point of the preceded window is dropped.

	Leontief		CES	
	ω	λ	ω	λ
h = 1	**0.13921**	**0.10817**	**0.13712**	**0.10706**
h = 2	**0.16750**	**0.19527**	**0.17581**	**0.18468**
h = 3	**0.19836**	**0.29957**	**0.21159**	**0.27826**
h = 4	**0.25784**	**0.42545**	**0.25080**	**0.39442**
h = 5	**0.32235**	**0.57081**	**0.31709**	**0.54217**
h = 6	**0.36468**	**0.72471**	**0.36028**	**0.66106**
h = 7	**0.41941**	**0.88183**	**0.39795**	**0.80009**
h = 8	**0.49479**	1.03970	**0.44024**	**0.93821**

Table 9: Relative performance of the model with break against a VAR.

Table 9 displays the new performance ratio with the explained methodology. Results are globally similar than the previous attempt with a growing sample although the VAR process do slightly better than previously for ω. Nonetheless, with respect to Section 5.2, while ratio of the error forecasts of the state variable for the employment rate λ show an upper ratio until horizon 7, it shows significant improvement on the remaining horizon especially for the CES specification. Finally, here a clear difference of performance in favor of the CES production function appears in table 9.

5.4 Understanding the Results for the Employment Rate

Throughout the backtesting, the employment rate, λ, did not perform as well as the other state variable ω. To understand why, one should focus on the performance and appreciate how the sub-periods have influenced the determination of the global measures. For the sake of clarity, one can consider the time horizon $h = 8$, in the column CES, and λ of table 8, 1.07286. This value is D_8^x / D_8^{VAR}, where x stands for the Goodwin model with a CES production function. The goal of the proposed exercise is to plot all components that made D_8^x and D_8^{VAR}, namely, $\forall i_8 = 1, \ldots, 100$, $d_\gamma(x_{i_8}^x, x_{i_8}^t)$, and $d_\gamma(x_{i_8}^{VAR}, x_{i_8}^t)$.

[35] This observation holds for the Lotka-Volterra-like model as well as the VAR.

[36] This frame 150 represents slightly more that half of the sample.

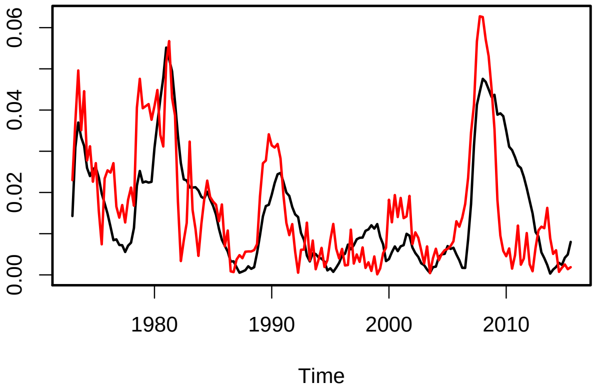

Figure 6: The performance of λ for the Goodwin model with a CES technology, in red, and for the VAR model, in black. Horizon: 8 quarters; estimated on a rolling window.

Figure 6 shows the timeseries $d_\gamma(x_{i_4}^{VAR}, x_{i_4}^t)$ in black and the timeseries $d_\gamma(x_{i_4}^x, x_{i_4}^t)$ in red. At any given date, when the red line is above the black line, it means that the error made by the VAR for a four-period-ahead forecast is lower than the error of the model x. Qualitatively, one sees that the VAR outperforms the model x for the last recession periods: (i) the oil shocks of the 1970s'; (ii) the Kuwait invasion (reported from 1990:Q3 to 1991:Q1.[37]); (iii) the burst of the dot-com bubble (reported from 2001:Q1 to 2001:Q4); and (iv) the subprime mortgage crisis (reported from 2007:Q4 to 2009:Q2). Removing those periods would certainly increase the accuracy of the forecast.

An empirical extension with the goal of increasing the reliability of the prediction made for λ could be the theoretical extension proposed by Keen (1995) and is a promising avenue for future research. Keen (1995) introduced debt into the dynamic, and by doing so, necessarily introduced the investment function that plays an important role in the global dynamics, especially for the dynamic of λ,

$$\dot{\lambda} = \lambda \left(\frac{\kappa(\pi)}{\nu} - [\alpha + \beta + \delta] \right),$$

where $\pi = 1 - \omega - rd$, with r as interest rate and d representing debt-to-GDP ratio. If one sets $\kappa(x) = x$ and $r = 0$, the equation for $\dot{\lambda}$ is the same as in system (1). By breaking the equality between profit and investment, Keen had a new dimension with debt. Although his dissipative model is three-dimensional, it gives room for emerging phenomenon such the Minsky moment and financial instability. Nonetheless, Keen's assumption allows for more flexibility on the global behavior of λ that may better capture the crisis effect. Note that van der ploeg mentioned this extension as an extension within the Goodwin framework.

6 Concluding Remarks and Further Extensions

This paper provided a global methodology to assess a class of macroeconomic models such as Goodwin's. I proposed a methodology to estimate continuous-time macroeconomic models with low-frequency data. An experiment was carried out by testing the

[37]The recession dates reported are from the NBER.

modern version of the Goodwin model and one of its extensions, the van der Ploeg (1985) model. To date, the results regarding the empirical success of the Goodwin model have been mixed. This paper tackled the question of the empirical estimation Goodwin model from a different perspective. Instead of inferring the model parameter by parameter or equation by equation, the author's approach allowed for an assessment of the model as a whole.

Results of the estimation show that the cycles could have structurally changed along the decades, meaning that, for the US, three sub-periods can be drawn out from the data: (i) 1948:Q1-1983:Q2; (ii) 1983:Q3-1999:Q3; and (iii) 1999:Q4-. For each of the sub-periods, the economy endowed with Leontief technology outperforms the CES counterpart in explaining the data's behavior. Additionally, a backtesting strategy based on an out-of-sample forecast was considered. The aim was to assess whether the model may be used for prospective scenarios. To do so, I compared its forecast ability against a VAR and obtained a globally positive result. That is to say that a global performance measure showed that the Goodwin-like models are–substantially–better at forecasting the state variables ω and λ, especially CES production function specification.

This paper added to a growing body of work that has developed theoretical models all based on the Goodwin-Lotka-Volterra model. The methodology used in the paper may been seen as a starting point for further empirical studies of extensions of Goodwin based models. A desirable extension would involve testing the Keen model and evaluating the investment function in order to provide a more accurate forecast for the employment rate, λ, and, in the meantime, a more precise estimate of the depreciation rate of capital. Furthermore, the estimation framework can be improved by extending the methodology to missing data points in order to allow the estimation technique to cope with various frequencies within the dataset.

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Appendices

Appendices are threefold: (i) compute the derivation of the stochastic models; (ii) introduce extensively the estimation technique named the simulated maximum likelihood (hereafter SMLE) illustrated by an example; and (iii) emphasize numerical problems of the inference of the short term Phillips curve.

A The models derivation

This appendix will introduce only the stochastic models. The deterministic counterparts will be deduce from the stochastic. The Goodwin-predator-prey model (Goodwin, 1967) will be first presented, the extension of van der Ploeg (1985) will follow.

A.1 The stochastic predator-prey model

Goodwin endows the productive sector with Leontief technology

$$Y_t = \min\left(a_t L_t, \frac{K_t}{\nu}\right),$$

where, Y_t, is the real output, L_t, is the employed population, a_t, the labor productivity, K_t, is the stock of capital, and ν is the (constant) capital-to-output ratio. By assuming full capacity utilization, the following equality holds:

$$Y_t = a_t L_t = \frac{K_t}{\nu}.$$

The labor productivity is assumed to grow according to a stochastic process

$$\frac{da_t}{a_t} = \alpha dt - \sigma_1(.)dB_t^1,$$

where $\sigma_1(.)$ is the diffusion function (the arguments can be all the state variables) and, B_t^1, is a Brownian motion. For the sake of clarity, the remaining of the model derivation is divided into two subsections, the first for the wage share and the second for the employment rate.

A.1.1 The wage share

Real wages growth is assumed to fluctuate according to the stochastic differential equation (hereafter SDEs)

$$\frac{dW_t}{W_t} = \Phi(\lambda_t)dt + \sigma_2(.)dB_t^2,$$

where $\Phi(.)$ is a smooth function, $\sigma_2(.)$ the diffusion process of, B_t^2, a Brownian motion assumed to be orthogonal to B_t^1. The wage share, ω, is defined as

$$\omega_t := \frac{W_t L_t}{Y_t} = \frac{W_t}{a_t}.$$

Using the multidimensional version of the îto lemma for the function $f(x, y) = x/y$, one has

$$\frac{d\omega_t}{\omega_t} = \left(\Phi(\lambda_t) - \alpha + \sigma_1^2(.)\right) dt + \sigma_1(.)dB_t^1 + \sigma_2(.)dB_t^2. \tag{7}$$

A.1.2 The employment rate

The total labor force, N, is assumed to grow exogenously so that

$$\frac{dN_t}{N_t} = \beta dt.$$

The employment rate, λ, is defined as

$$\lambda_t := \frac{L_t}{N_t}.$$

The capital accumulates according to

$$\frac{dK_t}{K_t} := \left(\frac{I_t}{K_t} - \delta\right) dt,$$

where, I_t, is the investment and δ is the (constant) depreciation rate of capital. Profits, Π, is defined as

$$\Pi_t := Y_t - W_t L_t$$

Thus, the profit rate (profit-to-output ratio) is defined by

$$\pi_t := \frac{\Pi_t}{Y_t} = 1 - \omega_t.$$

When assuming profits equal investment ($I_t = \Pi_t$),

$$\frac{dK_t}{K_t} = \left(\frac{(1-\omega_t)}{\nu} - \delta\right) dt.$$

The employment rate, λ, can be seen as

$$\lambda_t = \frac{L_t}{N_t} = \frac{K_t}{\nu a_t N_t}.$$

Using îto lemma for the function $f(x, y, z) = x/(\nu yz)$[38] one has

$$\frac{d\lambda_t}{\lambda_t} = \left[\frac{(1-\omega_t)}{\nu} - (\alpha + \beta + \delta) + \sigma_1^2(.)\right] dt + \sigma_1(.) dB_t^1. \tag{8}$$

Equations (7) and (8) make the two-dimensional stochastic prey-predator model. One can note that if $\sigma_1(.) = \sigma_2(.) = 0$, the system becomes the deterministic prey-predator model.

A.2 The stochastic van der Ploeg extension

Suppose that the productive sector is endowed with CES technology so that

$$Y := C\left[\pi K^{-\eta} + (1-\pi)(\lambda^L L)^{-\eta}\right]^{-\frac{1}{\eta}} \tag{9}$$

[38] Computation details will be a special case of the van der Ploeg model in the next Section.

where λ^L, the (CES) labor productivity, follows the stochastic process

$$\frac{\dot{\lambda}^L}{\lambda^L} = \alpha dt - \sigma_1 dB_t^1. \tag{10}$$

Assuming that the real wage, W, is determined as the marginal labor productivity of the CES production function,

$$\frac{\partial Y}{\partial L} = W. \tag{11}$$

For simplicity, one can consider that $L^e := \lambda^L L$, then

$$\frac{\partial Y}{\partial L^e} = \frac{\partial Y}{\partial L}\frac{1}{\lambda^L}. \tag{12}$$

Using equations (11) and (9),

$$\frac{\partial Y}{\partial L^e} = \frac{(1-\pi)}{C^\eta}\left(\frac{Y}{L^e}\right)^{1+\eta}.$$

Equalizing equations (11) and (12) through (9) :

$$\begin{aligned} \left(\frac{\omega}{1-\pi}\right)^{\frac{1}{\eta}} C &= \frac{Y}{L^e} \\ \Leftrightarrow \left(\frac{\omega}{1-\pi}\right)^{\frac{1}{\eta}} C\lambda^L &= \frac{Y}{L} \end{aligned} \tag{13}$$

A.2.1 The wage share

As previously, the wage bill-to-output ratio is $\omega = \frac{WL}{Y}$. Defining, $a := Y/L$, the labor productivity, the equality, $\omega = W/a$, holds. Until now, we do not know the dynamics of ω. The idea is to assume a SDEs for ω, so that

$$d\omega_t = \omega_t(f dt + g_1 dB_t^1 + g_2 dB_t^2). \tag{14}$$

One has to identify the f, g^1 and g^2 functions.[39] As previously, the real wage growth evolves according to

$$dW_t = W_t(\phi(\lambda_t)dt + \sigma_2 dB_t^2). \tag{15}$$

Or, from equation (13), one has

$$a_t = \left(\frac{\omega_t}{1-\pi}\right)^{\frac{1}{\eta}} C\lambda_t^L,$$

where ω is defined by equation (14) and λ^L by equation (10). Using the îto formula for the function $a = f(\omega, \lambda^L)$ one has

$$\frac{da_t}{a_t} = \frac{1}{\eta}\frac{d\omega_t}{\omega_t} + \frac{d\lambda_t^L}{\lambda_t^L} + \left(\frac{1-\eta}{\eta^2}\frac{(g_1^2+g_2^2)}{2} - \frac{1}{\eta}\sigma_1 g_1\right) dt. \tag{16}$$

[39]To simplify the notations, we will drop the "$(\cdot)$" for the functional form.

This result is obtained when assuming that the two Brownian motions, B_t^1, and B_t^2, are independent. Applying once again the îto formula for ω with equations (15) and (16)

$$d\omega_t = \omega_t \left(\frac{dW_t}{W_t} - \frac{da_t}{a_t} - \frac{d < a, W >_t}{a_t W_t} + \frac{d < a, a >_t}{a_t^2} \right). \tag{17}$$

Therefore, we can identify

$$\begin{aligned} \frac{d\omega_t}{\omega_t} &= \underbrace{\phi(\lambda_t) - \frac{f}{\eta} - \alpha - \frac{1}{2}\left(\frac{1-\eta}{\eta^2}(g_1^2 + g_2^2) - \frac{1}{\eta}\sigma_1 g_1 \right) - \frac{\sigma_2 g_2}{\eta} + \left(\frac{g_1}{\eta} - \sigma_1 \right)^2 + \left(\frac{g_2}{\eta} \right)^2}_{=f} dt \\ &+ \underbrace{\left(\sigma_1 - \frac{g_1}{\eta} \right)}_{=g_1} dB_t^1 + \underbrace{\left(\sigma_2 - \frac{g_2}{\eta} \right)}_{=g_2} dB_t^2. \end{aligned}$$

Thus,

$$\begin{aligned} g_1 &= \left(\frac{\eta}{\eta + 1} \right) \sigma_1 \\ g_2 &= \left(\frac{\eta}{\eta + 1} \right) \sigma_2 \\ f &= \left(\frac{\eta}{\eta + 1} \right) \left\{ \phi(\lambda_t) - \alpha - \frac{1}{2} \left(\frac{1-\eta}{(1+\eta)^2}(\sigma_1^2 + \sigma_2^2) - \frac{\sigma_1^2}{\eta + 1} \right) \right. \\ &- \left. \frac{\sigma_2^2}{\eta + 1} + \left(\frac{\sigma_1 \eta}{1 + \eta} \right)^2 + \left(\frac{\sigma_2}{1 + \eta} \right)^2 \right\}. \end{aligned}$$

Finally, the wage share dynamics is

$$\begin{aligned} \frac{d\omega_t}{\omega_t} &= \left(\frac{\eta}{\eta + 1} \right) \left\{ \phi(\lambda_t) - \alpha - \frac{1}{2} \left(\frac{1-\eta}{(1+\eta)^2}(\sigma_1^2 + \sigma_2^2) - \frac{\sigma_1^2}{\eta + 1} \right) \frac{\sigma_2^2}{\eta + 1} + \left(\frac{\sigma_1 \eta}{1 + \eta} \right)^2 + \left(\frac{\sigma_2}{1 + \eta} \right)^2 \right\} dt \\ &+ \left(\frac{\eta}{\eta + 1} \right) \sigma_1 dB_t^1 + \left(\frac{\eta}{\eta + 1} \right) \sigma_2 dB_t^2 \end{aligned} \tag{18}$$

One can note that if $\eta \to +\infty$, capital and labor do not substitute, i.e. we retrieve the Leontief case as previously defined,

$$\frac{d\omega_t}{\omega_t} = \left(\phi(\lambda_t) - \alpha + \sigma_1^2 \right) dt + \sigma_1 dB_t^1 + \sigma_2 dB_t^2.$$

A.2.2 The employment rate

As previously mentioned, the total workforce grows at a yearly pace β

$$\frac{dN_t}{N_t} = \beta dt$$

The employment rate is defined by, $\lambda_t := L_t / N_t$, the following dynamic holds

$$\frac{d\lambda_t}{\lambda_t} = \frac{dL_t}{L_t} - \frac{dN_t}{N_t}.$$

This is a consequence of equation (17) and the fact that N, the total labor force, is deterministic. The capital accumulates so that

$$\frac{dK_t}{K_t} := \left(\frac{(1-\omega_t)}{\nu_t} - \delta\right) dt. \tag{19}$$

Using equations (9) and (11), the capital-to-output ratio is such that

$$\nu_t = \left(\frac{1-\omega_t}{\pi}\right)^{-\frac{1}{\eta}} \frac{1}{C}.$$

Thus, the accumulation of capital can be written

$$\frac{dK_t}{K_t} := \left(C\pi^{-1/\eta}(1-\omega_t)^{1+1/\eta} - \delta\right) dt. \tag{20}$$

Using the îto lemma for the function $f(\omega_t) = \nu_t$, the capital-to-output ratio evolves so that

$$\frac{d\nu_t}{\nu_t} = \frac{1}{\eta}\frac{\omega_t}{1-\omega_t}\frac{d\omega_t}{\omega_t} + \left(\frac{\omega_t}{1-\omega_t}\right)^2 \left(\frac{1}{1+\eta}\right)\left(\frac{\sigma_1^2+\sigma_2^2}{2}\right) dt. \tag{21}$$

Again, by using the îto formula for $L = K/(\nu a)$–the function will be $L = f(K,\nu,a) = K/(\nu a)$. One has

$$\begin{aligned} \frac{dL}{L} &= \frac{dK}{K} - \frac{d\nu}{\nu} - \frac{da}{a} - \frac{d<K,a>_t}{Ka} - \frac{d<K,\nu>_t}{K\nu} + \frac{d<\nu,a>_t}{\nu a} \\ &+ \frac{d<a,a>_t}{aa} + \frac{d<\nu,\nu>_t}{\nu\nu} + \frac{d<K,K>_t}{KK}, \end{aligned}$$

thus,

$$\begin{aligned} \frac{dL_t}{L_t} &= \left(C\pi^{-1/\eta}(1-\omega_t)^{1+1/\eta} - \delta\right) dt \\ &- \frac{1}{\eta}\frac{\omega_t}{1-\omega_t}\frac{d\omega_t}{\omega_t} - \left(\frac{\omega_t}{1-\omega_t}\right)^2\left(\frac{1}{1+\eta}\right)\left(\frac{\sigma_1^2+\sigma_2^2}{2}\right) dt \\ &- \frac{1}{\eta}\frac{d\omega_t}{\omega_t} - \alpha + \sigma_1 dB_t^1 - \left(\frac{1-\eta}{(1+\eta)^2}\frac{(\sigma_1^2+\sigma_2^2)}{2} - \frac{\sigma_1^2}{\eta+1}\right) dt \\ &- 0 - 0 + \left(\frac{\omega_t}{1-\omega_t}\right)\left\{-\eta\left(\frac{\sigma_1}{1+\eta}\right)^2 + \left(\frac{\sigma_2}{1+\eta}\right)^2\right\} dt \\ &+ \left(\frac{\eta}{\eta+1}\right)^2 \sigma_1^2 dt + \left(\frac{\sigma_2}{1+\eta}\right)^2 dt \\ &+ \left[\left(\frac{\omega_t}{1-\omega_t}\frac{1}{1+\eta}\right)\right]^2 (\sigma_1^2+\sigma_2^2)dt + 0. \end{aligned}$$

The employment rate's SDEs is

$$
\begin{aligned}
\frac{d\lambda_t}{\lambda_t} &= \left(C\pi^{-1/\eta}(1-\omega_t)^{1+1/\eta} - (\delta+\beta+\alpha)\right) dt \\
&- \left(\frac{\omega_t}{1-\omega_t}\right)^2 \left(\frac{1}{1+\eta}\right) \left(\frac{\sigma_1^2+\sigma_2^2}{2}\right) dt \\
&- \left(\frac{1-\eta}{(1+\eta)^2}\frac{(\sigma_1^2+\sigma_2^2)}{2} - \frac{\sigma_1^2}{\eta+1}\right) dt \\
&+ \left(\frac{\omega_t}{1-\omega_t}\right) \left\{-\eta\left(\frac{\sigma_1}{1+\eta}\right)^2 + \left(\frac{\sigma_2}{1+\eta}\right)^2\right\} dt \\
&+ \left(\frac{\eta}{\eta+1}\right)^2 \sigma_1^2 dt + \left(\frac{\sigma_2}{1+\eta}\right)^2 dt \\
&+ \left[\left(\frac{\omega_t}{1-\omega_t}\frac{1}{1+\eta}\right)\right]^2 (\sigma_1^2+\sigma_2^2)dt - \frac{1}{\eta}\left(\frac{d\omega_t}{\omega_t(1-\omega_t)}\right) + \sigma_1 dB_t^1
\end{aligned}
\tag{22}
$$

The stochastic van der Ploeg model is entirely defined by equations (22) and (18). It is worth mentioning that if $\eta \to +\infty$ and $C := 1/\nu$ where ν is the constant capital-output ratio, one has

$$\frac{d\lambda_t}{\lambda_t} = \left(\frac{1-\omega_t}{\nu} - (\alpha+\beta+\delta) + \sigma_1^2\right) dt + \sigma_1 dB_t^1$$

Furthermore, if the model is deterministic ($\sigma_1 = \sigma_2 = 0$), we retrieve a model close to the one of Grasselli & Maheshwari (2017a), i.e. :

$$\frac{d\lambda_t}{\lambda_t} = \left(C\pi^{-1/\eta}(1-\omega_t)^{1+1/\eta} - (\delta+\beta+\gamma) - \frac{1}{\eta}\left(\frac{d\omega_t}{\omega_t(1-\omega_t)}\right)\right) dt.$$

B The SMLE method

The appendix will derive in length the SMLE method in a multivariate framework. Most of this derivation is available in Durham & Gallant (2002) in an univariate framework.

B.1 Notations

In what follows, one can consider a multivariate SDEs, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ of the form

$$dX_t = f(X_t)dt + g(X_t)dB_t, \tag{23}$$

where

- $X_t \in \mathbb{R}^n$,
- B_t is a d-dimensional Brownian motion,
- $f : \mathbb{R}^n \to \mathbb{R}^n$,
- $g : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, where $\forall x$, $g^{-1}(x)g(x)$ is positive definite.

B.2 Numerical Simulation of the Solution

Analytic solution of economic models are unlikely to be available. To approximate the numerical solution of model (23), one can use the generalization of the Euler explicit method for ordinary differential equations to stochastic differential equations, namely the Euler-Maruyama scheme. For SDEs, there exists several manners of approximating the solution, for instance the Jimenez et al. (1999) scheme. Despite its low efficiency, the Euler, the Euler-Maruyama scheme is computationally stable for any case.

Consider the model (23), with the initial condition, $X_0 = x_o$, and suppose that one wishes to solve the SDEs on some interval of time $[0, T]$. Then the Euler–Maruyama approximation to the true solution X is the Markov chain Y defined in the following manner:

- Consider a partition of the interval $[0, T]$ into N equal subintervals of width $\Delta t > 0$:
$$0 = \tau_0 < \tau_1 < \cdots < \tau_N = T \text{ and } \Delta t = T/N.$$
- Set the initial condition $Y0 = x0$.
- Recursively one can define Y_n, for $1 \leq n \leq N$, by
$$Y_{n+1} = Y_n + f(Y_n)\,\Delta t + g(Y_n)\,\Delta W_n, \tag{24}$$
where
$$\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}.$$
The random variables ΔW_n are independent and identically distributed normal random variables with expected value zero and variance Δt.

B.3 The estimation procedure

The methodology and the notations are borrowed from Durham & Gallant (2002), and is extended to the multivariate analysis.

B.3.1 Overview

If one writes the joint likelihood function as being $p(x_1, \ldots, x_T)$, where the observations are $x_i \in R^n, \forall i \in \{1, \ldots, T\}$, one can rewrite the likelihood function as being:[40]

$$p(x_1, \ldots, x_T) = p(x_1) \prod_{i=2}^{T} p(x_i, i; x_{i-1}, i-1).$$

The objective of the SMLE procedure is to give methodology to compute $p(x_t, t; x_s, s)$, in other words the transition probability of the process, x, from time s to time t. The first order approximation $p^{(1)}(x_t, t; x_s, s)$ defined by (24) will be accurate if the interval $[s, t]$ is short enough. Otherwise, one may partition the interval such that the first-order approximation is sufficiently accurate on each subinterval ($s = \tau_0 < \ldots < \tau_M = t$).

[40] The first element of the likelihood, $p(x_1)$, is unknown and will be neglected in the computation of the likelihood.

The random variables, x_{τ_i}, are unobserved, and must be integrated out. Because the process is Markovian, one obtains

$$\begin{aligned} p(x_t, t; x_s, s) &\approx p^{(M)}(x_t, t; x_s, s) \\ &:= \int \prod_{m=0}^{M-1} p^{(1)}(u_{m+1}, \tau_{m+1}; u_m, \tau_m) d\lambda(u_1, \ldots, u_{M-1}) \end{aligned}$$

where, λ, is here the Lebesgue measure, and the conventions $u_0 = x_s$, and $u_M = x_t$ are used. Monte Carlo integration is generally the only feasible way to evaluate the integral.

For $s < t$ suppose that $x_t|x_s$ has a transition density $p(x_t, t; x_s, s)$ and let

$$p^{(1)}(x_t, t; x_s, s) = \phi(x_t; x_s + f(x_s)(t-s), g(x_s)\sqrt{(t-s)})$$

where $\phi(x, f, g)$ is the Gaussian density, be its first-order approximation. One can prove that, under mild assumptions[41] reported in Durham & Gallant (2002),

$$\lim_{M \to +\infty} p^{(M)} p(., t; x_s, s, \theta) = p(., t; x_s, s, \theta) \text{ , in } L^1(\lambda) \tag{25}$$

Pedersen (1995a,b) show that the convergence presented above is reach for the linear case. To the best of my knowledge, no proof has be made with nonlinear functions nor counterexample as been found. In the paper, each time a nonlinear form is used, it has been tested using some data generating process (hereafter DGP).

B.3.2 How to Compute the Integral?

Let $\{u_k = (u_{k,1}, \ldots, u_{k,M-1}), k = 1, \ldots, K\}$ be independent draws from q–an importance sampler. One can define

$$p^{(M,K)}(x_t, t; x_s, s, \theta) = \frac{1}{K} \sum_{k=1}^{K} \frac{\prod_{m=1}^{M} p^{(1)}(u_{k,m}, \tau_m; u_{k,m-1}, \tau_{m-1}, \theta)}{q(u_{k,1}, \ldots, u_{k,M-1})} \tag{26}$$

where $u_{k,0} = x_s$ and $u_{k,m} = x_t$ for all k. Under some mild assumptions and the strong law of large numbers, on has

$$\lim_{K \to +\infty} |p^{(M,K)}(x_t, t; x_s, s, \theta) - p^{(M)}(x_t, t; x_s, s, \theta)| = 0 \text{ a.s.}$$

Durham & Gallant (2002) remarks that when M increases, for a fix K, the bias will be reduced but the variance will increase. One may increase sufficiently K in order to reduce that variance but it is costly since the variance decreases at the speed $1/\sqrt{K}$.

B.3.3 Which Importance Sampler to choose?

The importance sampler that will be used is the one which draws u_{m+1} from a Gaussian density based on the first approximation conditional on u_m and x_t. That is, treating u_m and $u_M = x_t$ as fixed, one draws u_{m+1} from the density

$$\begin{aligned} p(u_{m+1}|u_m, u_M) &= p(u_{m+1}|u_m) p(u_M|u_{m+1}) / p(u_M|u_m) \\ &= \phi(u_{m+1}; u_m + \tilde{\mu}_m \delta, \tilde{g}(\cdot)_m^2 \delta) \end{aligned}$$

[41] Including a nonexploding, unique weak solution of (5).

where $\delta = (t-s)/M$, and

$$\tilde{\mu}_m = \left(\frac{u_M - u_m}{t - \tau_m}\right), \ \tilde{g}(\cdot)_m = \left(\frac{M-m-1}{M-m}\right) g(\cdot)^2$$

This sampler is called the modified Brownian bridge.[42]

Although it is possible to compute the likelihood directly from (26), it is time-consuming. Suppose we have data generated, on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, by the process

$$dX = f(X)dt + g(X)dB^{\mathbb{P}} \tag{27}$$

where B is a $d-$dimensional Brownian motion under the probability $\mathbb{P}$. Suppose we want to change the drift to the process by including $\gamma = \tilde{\mu}(X) - f(X)$ so that the drift becomes $\tilde{\mu}(X)$. Provided that $\gamma_t(X_t)$ is adapted to B_t and there is an adapted solution u to the equation

$$u(X) = g(X)^{-1}(\tilde{\mu}(X) - f(X)) \tag{28}$$

then the process can be rewritten as

$$dX = \tilde{\mu}(X)dt + g(X)[u(X)dt + dB_t^{\mathbb{P}}]$$

under $\mathbb{P}$. The process will also satisfy

$$d\tilde{X} = \tilde{\mu}(\tilde{X})dt + g(\tilde{X})d\tilde{B}_t^{\mathbb{Q}} \tag{29}$$

Assuming weak uniqueness, the solution of the process (27) as the same distribution than the process in (29). Girsanov's theorem tells us that the Radon-Nykodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ is

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} &= M_t \\ &= exp\left\{-\sum_{i=1}^{d}\int_0^t u^{(i)}(X_s)dB_s^{(i)} - \frac{1}{2}\int_0^t \|u(X_s)\|^2 ds\right\} \end{aligned}$$

or written differently, under $\mathbb{P}$

$$dM_t = M_t\left(\sum_{i=1}^{d} -u^{(i)}(X_s)dB_s^{(i)}\right)$$

or under $\mathbb{Q}$,

$$dM_t = M_t\left(\sum_{i=1}^{d} u^{(i)}(X_s)d\tilde{B}_s^{(i)}\right)$$

with the initial condition that $M_s = 1$ and where $u^{(i)}(X_s)$ refers the the i^{th} coordinate of (28). Thus one can obtain the continuous-time expression

$$p(x_t, t; x_s, s) = \int p(x_t, t; u, \tau_{M-1})\rho_{M-1}(u)dQ_{M-1}(u), \tag{30}$$

[42]It is named after Durham & Gallant (2002).

where Q_{M-1} is the probability measure induced by $\tilde{X}_{\tau_{M-1}}$. The integral is computed by generating samples $\{(u_{k,M-1}, r_{k,M-1})\}$ from the joint process $(\tilde{X}^{(M)}_{M-1}, M^{(M)}_{M-1})$ using the Euler-Maruyama scheme,

$$p^{(M,K)}(x_t, t; x_s, s, \theta) = \frac{1}{K}\sum_{k=1}^{K} p^{(1)}(x_t, t; u_{k,M-1}, \tau_{M-1}) r_{k,M-1}.$$

Durham & Gallant (2002) found that it is more stable to base the Euler-Maruyama scheme for M on

$$d(\log(M)) = -\frac{1}{2}\sum_{k=1}^{d}(u^{(i)}(\tilde{X}))^2 dt + \sum_{k=1}^{d} u^{(i)}(\tilde{X}) d\tilde{B} \tag{31}$$

Finally, I will compute the simulated $\log$-likelihood

$$l_n^{(M,K)}(\theta) = \sum_{i=1}^{n} \log p^{(M,K)}(X_i, t_i; X_{i-1}, t_{i-1}, \theta).$$

C Example of the estimation method with a DGP

Simulation of 200 time steps of the Goodwin-like models give the following dynamics, The starting value of the simulation are $(\omega_0, \lambda_0) = (0.62, 0.92)$.

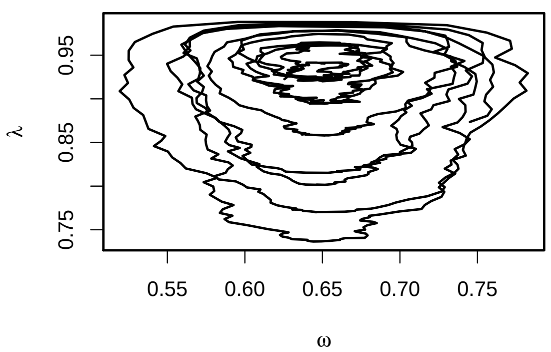

Figure 7: Simulation: The phase portrait of the system 3.

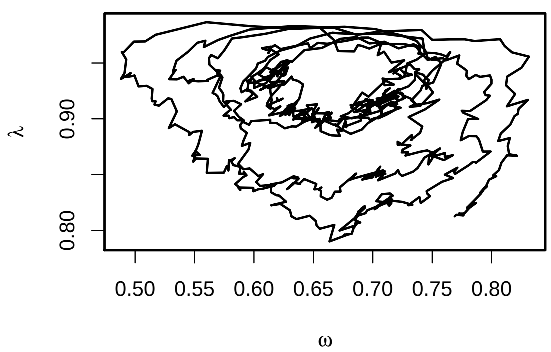

Figure 8: Simulation: The phase portrait of the system (4)

C.1 Test of the estimation procedure

For the estimation, I use $M = 16$ and $K = 128$.[43] Regarding the results of table 10, one can note that we cannot rely on the sign of the sigmas since the Brownian motion is symmetric[44] and the model specification is linear with respect to the Brownian motion.

[43] These parameters are chosen to make the tradeoff between time of computation and accuracy of the results.

[44] Indeed, if W_t is a Brownian motion, $-W_t$ is also a Brownian motion.

Parameter	Starting Value	Estimated Value	True Parameter
ϕ_0	0.015	1.9893e-02	1.7806e-02
ϕ_1	0.0001	6.4706e-05	6.4103e-05
ψ_0	2.7	3.0245	3
ψ_1	0.1	0.11580	1.1667e-01
σ_1	0.02	-1.3836e-03	0.0015
σ_2	0.02	1.4382e-02	0.015

Table 10: Results for the estimation of the simulated system (3). $M = 16$, $K = 128$.

Parameter	Starting Value	Estimated Value	True Parameter
ϕ_0	0.015	1.7224e-02	1.7806e-02
ϕ_1	0.0001	6.8359e-05	6.4103e-05
C_b	0.1	1.2442e-01	0.12
ψ_1	0.1	4.1046e-02	4.0100e-02
η	0.1	5.9692e+01	100
σ_1	0.01	1.0325e-02	1.5000e-02
σ_2	0.01	1.4528e-02	1.5000e-02

Table 11: Results for the estimation of the simulated system (4). $M = 16$, $K = 128$.

D Numerical test on the inference of the short Term Phillips curve

This appendix aims to show how a linear regression techniques to estimate a continuous-time short term linear Phillips curve of the form

$$\frac{\dot{W}}{W} = \Phi(\lambda)$$

can lead to spurious results. For that purpose, we generate data for the employment rate, λ. Suppose that λ is generated according to an autoregressive process with a lag one, so that

$$\lambda_t = 0.96 * (1 - 0.5^{1/(N/T/4)}) + 0.5^{1/(N/T/4)}\lambda_{t-1/N} + \varepsilon_t,$$

where $\varepsilon_t \sim \mathcal{N}(0, T/N)$. The parameters are chosen so that the mean for λ is 0.96 and the correlation of λ_t and $\lambda_{t+1/4}$ is 0.5. $N = 50 \times 10000$ is the number of subperiods between 0 and $T = 50$. Using that λ, the wages will be simulated using the Euler-Maruyama scheme of the stochastic differential process

$$dW_t = W_t \left((\phi(\lambda_t))dt + \sigma dB_t \right).$$

With $W_0 = 100$, and $\sigma = 0.01$. The $\Phi(.)$ function is suppose to be linear so that

$$\begin{aligned} \Phi(\lambda) &= \phi_0 \times \lambda + \phi_1 \\ &= 0.89 \times \lambda - 0.82 \end{aligned}$$

Two samples will be created by talking the the value that correspond to one quarter for both λ and W. The $\log -$return of the quarterly timeserie of W is computed, its scatterplot with the quarterly timeserie of λ. If one uses linear regression techniques

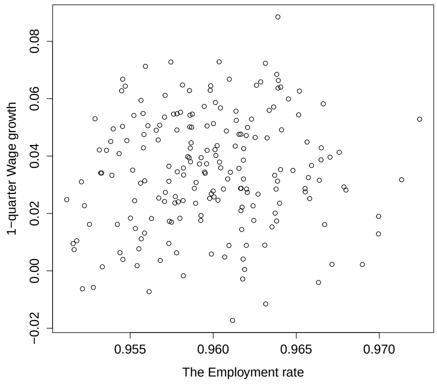

Figure 9: Simulation: The employment rate versus the one-quarter wage growth.

on the quarterly data to estimate the model of the DGP given above, one finds

$$\hat{\phi}_0 = 0.371;\ \hat{\phi}_1 = -0.3214.$$

One can conclude that, in this example, the results are at odds with the parameters used for the DGP.

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